Merewether High School

2010

Higher School Certificate

Trial Examination

Mathematics

Extension 2

General Instructions

- Reading time - 5 minutes
- Working time – 3 hours
- Board-approved calculators may be used
- Write using black or blue pen
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Write your student number at the top of every page
- Start each question on a new sheet of paper
- Each question is to be handed in separately

Total marks – 120

- Attempt Questions 1 – 8
- All questions are of equal value

This paper MUST NOT be removed from the examination room

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Marks

Question 1 **15 Marks** **Start a new page**

a.) Find $\int \frac{dx}{x.lnx}$ [2]

b.) Find $\int \frac{\cos x}{\sqrt{4-sin^2 x}} dx$ [3]

c.) Evaluate in simplest exact form $\int_0^{\frac{\pi}{3}} \frac{\sec x+\tan x}{\cos x}\, dx$ [3]

d.) Using partial fractions evaluate $\int_1^2 \frac{3}{3x-x^2} dx$ [3]

e.) Evaluate $\int_{-1}^{0} \frac{(x-1)}{x^2+2x+2} dx$ [4]

Question 2 **15 Marks** **Start a new page**

a.) The complex number z is given by $z = 1 + \frac{1+i}{1-i}$ find:

i. $Re(z)$ [1]

ii. $Im(z)$ [1]

iii. $|z|$ [1]

iv. $\arg(z)$ [1]

b.) On separate Argand diagrams, draw neat labelled sketches for each of the regions:

i. $|z-2-i| = 4$ [1]

ii. $Re(z+iz) \geq 2$ [2]

c.) i. Determine the locus of the complex number z given:
$\arg(z-2) = \frac{\pi}{4} + \arg(z+2)$ [2]

ii. sketch the locus on the Argand diagram [2]

d.) i. Express $2\sqrt{3} - 2i$ in polar form [1]

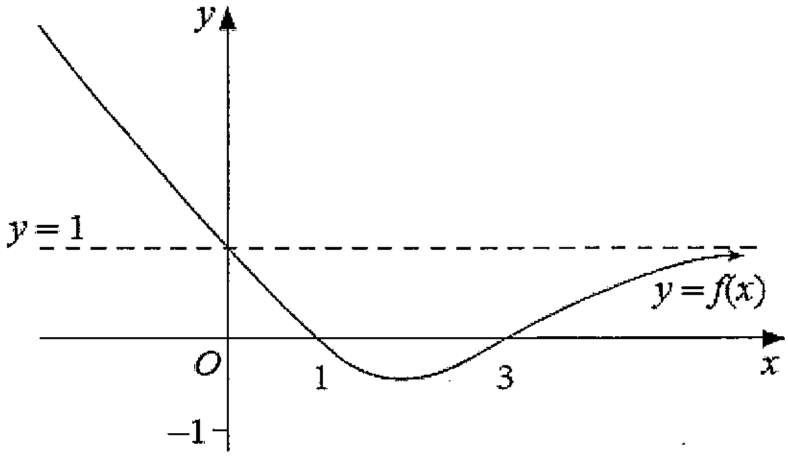
ii. Hence or otherwise evaluate $(2\sqrt{3} - 2i)^7$ giving your answer as a complex number in Cartesian form [3]

Marks

Question 3 **15 Marks** **Start a new page**

a.) The polynomial $P(x) = x^3 - 12x^2 + 36x + c$ has a double zero. Find any possible values of the real number c. [3]

b.)



The diagram shows the graph of the function $y = f(x)$.
The function has a horizontal asymptote as $y = 1$.
Draw separate half-page sketches of the graphs of the following functions:

i. $y = |f(x)|$ [2]

ii. $y = \frac{1}{f(x)}$ [2]

iii. $y = \ln[f(x)]$ [2]

c.) $P(x) = x^4 - 2x^3 + 4x^2 - 3x + 1$ and the equation $P(x) = 0$ has roots $\alpha, \beta, \gamma,$ and δ.

i. Show that the equation $P(x) = 0$ has no integer roots [1]

ii. Show that $P(x) = 0$ has a real root between *0* and *1* [1]

iii. Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -4$ [2]

iv. Hence find the number of real roots of the equation $P(x) = 0$, giving reasons [2]

Marks

Question 4 **15 Marks** **Start a new page**

a.) For the curve $y^3 + 2xy + x^2 + 2 = 0$

i. Show that $\frac{dy}{dx} = \frac{-2(y+x)}{3y^2+2x}$ [3]

ii. Find the coordinates of any stationary points on the curve [2]

b.) Points $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\cos\varphi, b\sin\varphi)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

i. Find the equation of the chord PQ [1]

ii. Hence show that if PQ subtends a right angle at the point $A(a, 0)$then PQ passes through a fixed point $T(t, 0)$ on the x-axis, where $t = \frac{ae^2}{2te^2}$ [4]

c.) Find $\int \sin^{-1} x\, dx$ [2]

d.) The point A represents the complex number z_3 and the point Z_1represents the complex number z_1. The point Z_1 is rotated about A through a right angle in the positive direction to take up the position Z_2, representing the complex number z_2. Show that $z_2 = (1 - i)z_3 + iz_1$ [3]

Question 5 **15 Marks** **Start a new page**

a.) Use Mathematical Induction to show that
$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$
for all positive integers $n \geq 2$ [4]

b.) The area below the curve $y = bx - ax^2$ (where $a > 0$ and $b > 0$) and above the *x*-axis is rotated about the *y* axis through a complete revolution.

Show, using a slice technique or otherwise, that the volume of the solid so formed is:

$\frac{\pi b^4}{6a^3}$ cubic units [3]

Marks

c.) Patrick's accuracy with darts is such that if he scores a bullseye, the probability of doing the same on the next throw is $\frac{2}{3}$, however if he misses, the probability that he again misses the bullseye on the next throw is $\frac{3}{4}$.
The probability of hitting the bullseye on the first throw is $\frac{1}{3}$.

i. What is the probability that he throws a bullseye on the second throw? [2]

ii. What is the probability that he misses a bullseye on the third throw? [2]

d.) On a certain day, the depth of water in a harbour at high tide at 5 am is 9 m. At the following low tide at 11:20 am the depth is 3 m. Assuming that the tidal motion is simple harmonic, find the latest time before noon that a ship can enter the harbour if a minimum depth of 7.5 m of water is required. [4]

Question 6 **15 Marks** **Start a new page**

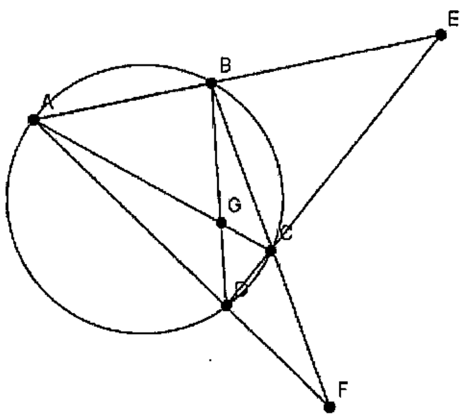
a.) For the hyperbola $\frac{x^2}{16} - \frac{y^2}{128} = 1$ find:

i. The eccentricity [1]

ii. The coordinates of the foci [1]

iii. The equations of the directrices [1]

b.) ABCD is a cyclic quadrilateral whose opposite sides meet at E and F and whose diagonals meet at G.



i. If BD bisects the angles at B and D, prove that $\angle$BAD is a right angle. [3]

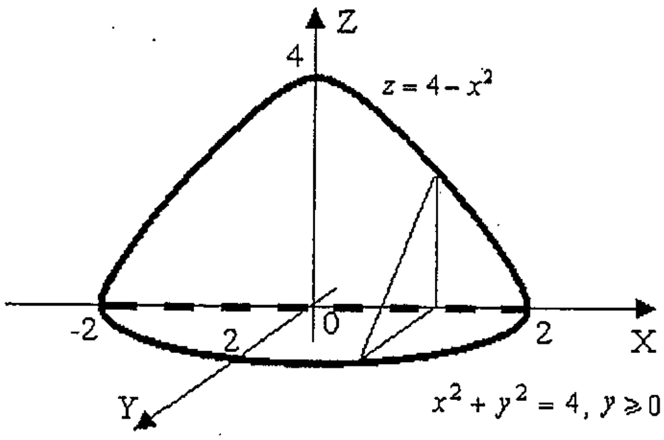
ii. What is the relation between the angles EAF and ECF? [3]

c.) By taking strips parallel to the axis of rotation, use the method of cylindrical shells to find the volume of the solid obtained by rotating the region $\{(x, y): 0 \leq y \leq 2x - x^2\}$ about the y-axis. [3]

d.) By using two applications of integration by parts, evaluate $\int_0^{\frac{\pi}{2}} e^x \cos x \, dx$ [3]

Question 7 **15 Marks** **Start a new page**

a.) The solid shown has a semicircular base of radius 2 units. Vertical cross-sections perpendicular to the diameter are right-angled triangles whose height is bounded by the parabola $z = 4 - x^2$.



i. By slicing at right angles to the x-axis, show *that the volum*e of the solid is given by $V = \int_0^2 (4 - x^2)^{3/2} \, dx$, [3]

ii. and hence calculate this volume. [3]

b.) For positive real numbers $a, b, c, a_1, a_2, \cdots, a_n$:

i. Show that $a + \frac{1}{a} \geq 2$ [1]

ii. Hence show that $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$ and $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9$ [2]

iii. Show that $(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2$ [2]

c.) $P\left(cp, \frac{c}{p}\right), Q\left(cq, \frac{c}{q}\right), R\left(cr, \frac{c}{r}\right)$ are three points on the rectangular hyperbola $xy = c^2$ such that the parameters p, q, r are in geometric progression.

i. Explain why P and R must lie on the same branch of the hyperbola. Under what condition will Q lie on the opposite branch to P and R? [1]

ii. Show that the chord PR is parallel to the tangent to the hyperbola at Q. [3]

Question 8 15 Marks Start a new page

a.) $I_n = \int_0^1 ln(1+x)\,dx, n = 0, 1, 2, \cdots$

i. Show that $\int ln(1+x)\,dx = (1+x)ln(1+x) - x + c$ [1]

ii. Show that $(n+1)I_n = 2ln2 - \frac{1}{n+1} - nI_{n-1},\ \ n = 1, 2, \cdots$ [2]

iii. Evaluate $3I_2$ and $4I_3$ [2]

iv. Show that $(n+1)I_n = \begin{cases} \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{n+1}, & n\ odd \\ 2ln2 - \left(\frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{n+1}\right), & n\ even \end{cases}$ [2]

b.) i. If $t = \tan x$ prove that $\tan 4x = \frac{4t(1-t^2)}{t^4 - 6t^2 + 1}$ [2]

ii. If $\tan x \tan 4x = 1$ deduce that $5t^4 - 10t^2 + 1 = 0$ [1]

iii. Prove that $x = 18°$ and $x = 54°$ satisfy the equation $\tan x \tan 4x = 1$ [2]

iv. Deduce that $\tan 54° = \sqrt{\frac{5+2\sqrt{5}}{5}}$ [3]

End of paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Extension 2 Trial 2010

1) a) $\int \frac{dx}{x \ln x} = I \qquad$ Let $u = \ln x$

$du = \frac{1}{x} \cdot dx$ (1)

$I = \int \frac{du}{u}$

$= \ln u + C$

$= \ln(\ln x) + C \quad (1)$

b) $\int \frac{\cos x}{\sqrt{4 - \sin^2 x}} \, dx = I$

Let $2y = \sin x$

$2dy = \cos x \, dx$

$I = \int \frac{2dy}{\sqrt{4 - 4y^2}} \quad (1)$

$= \int \frac{dy}{\sqrt{1 - y^2}}$

$= \sin^{-1} y + C \quad (1)$

$= \sin^{-1}\left(\frac{\sin x}{2}\right) + C \quad (1)$

c) $\int_0^{\frac{\pi}{3}} \frac{\sec x + \tan x}{\cos x} \, dx$

$= \int_0^{\frac{\pi}{3}} (\sec^2 x + \sec x \tan x) \, dx \; (1)$

$= \left[\tan x + \sec x\right]_0^{\frac{\pi}{3}} \quad (1)$

$= (\sqrt{3} + 2) - (0 + 1)$

$= \sqrt{3} + 1 \quad (1)$

d) $I = \int_1^2 \frac{3}{3x - x^2} \, dx$

$\frac{3}{3x - x^2} \equiv \frac{A}{3 - x} + \frac{B}{x}, \quad x \neq 0, 3$

$3 = Ax + 3B - Bx$

$3 = (A - B)x + 3B$

$\therefore B = 1$

$A - B = 0$

$\therefore A = 1 \quad (1)$

$I = \int_1^2 \frac{1}{3 - x} + \frac{1}{x} \, dx$

$= \left[-\ln|3 - x| + \ln|x|\right]_1^2 \quad (1)$

$= (-\ln(1) + \ln 2) - (-\ln(2) + \ln($

$= 2\ln(2) \quad (1)$

e) $I = \int_{-1}^0 \frac{(x - 1)}{x^2 + 2x + 2} \, dx$

$= \frac{1}{2} \int_{-1}^0 \frac{(2x - 2)}{x^2 + 2x + 2} \, dx$

$= \frac{1}{2} \int_{-1}^0 \frac{2x + 2}{x^2 + 2x + 2} \, dx - \frac{1}{2} \int_{-1}^0 \frac{4}{x^2 + 2x + 2} \, dx \; (1$

$= \frac{1}{2} \left[\ln|x^2 + 2x + 2|\right]_{-1}^0 - \int_{-1}^0 \frac{2}{(x-1)^2 + 1} \, dx \; (1$

$= \frac{1}{2}(\ln 2 - \ln 1) - 2\left[\tan^{-1}(x + 1)\right]_{-1}^0 \; (1$

$= \frac{1}{2} \ln 2 - 2(\tan^{-1} 1 - \tan^{-1} 0)$

$= \frac{1}{2} \ln 2 - 2\left(\frac{\pi}{4} - 0\right)$

$= \frac{1}{2} \ln 2 - \frac{\pi}{2} \quad (1)$

Question 2.

a) $Z = 1 + \dfrac{1+i}{1-i}$

$= 1 + \dfrac{(1+i)^2}{(1-i)(1+i)}$

$= \dfrac{1+2i-1}{2}$

$= 1+i$

i) $Re(Z) = 1$ (1)

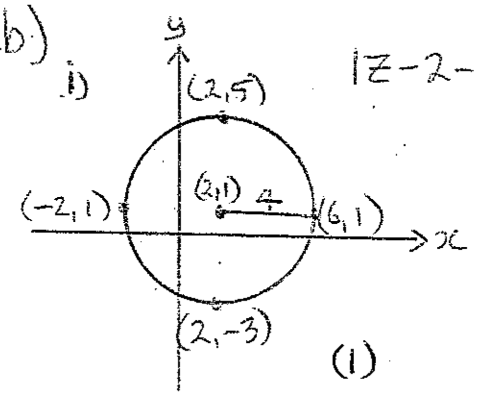
ii) $Im(Z) = 1$ (1)

iii) $|Z| = \sqrt{2}$ (1)

iv) $\arg Z = \tan^{-1}\dfrac{1}{1} = \dfrac{\pi}{4}$ (1)

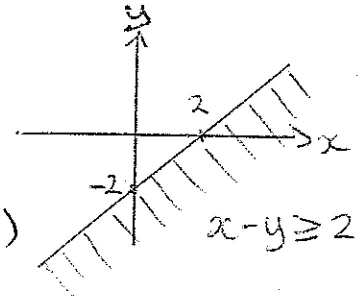
b)

i) $|Z-2-i| = 4$



(1)

ii.)



(1) $x - y \geq 2$

$Z + iZ$
$= x + iy + i(x+iy)$
$= x - y + i(x+y)$
$\therefore x - y \geq 2$
(1)

$Re(Z + iZ) \geq 2$

c) i) $\arg(z-2) = \dfrac{\pi}{4} + \arg(z+2)$

Let $\arg(z-2) = \theta$
and $\arg(z+2) = \phi$

$\theta = \dfrac{\pi}{4} + \phi$

$\tan\theta = \tan\left(\dfrac{\pi}{4} + \phi\right)$

$= \dfrac{\tan\frac{\pi}{4} + \tan\phi}{1 - \tan\frac{\pi}{4}\tan\phi}$

$= \dfrac{1+\tan\phi}{1-\tan\phi}$ (1)

$z - 2 = (x-2) + iy$

$\therefore \tan\theta = \dfrac{y}{x-2}$

and

$z + 2 = (x+2) + iy$

$\therefore \tan\phi = \dfrac{y}{x+2}$

$\therefore \dfrac{y}{x-2} = \dfrac{1 + \frac{y}{x+2}}{1 - \frac{y}{x+2}}$

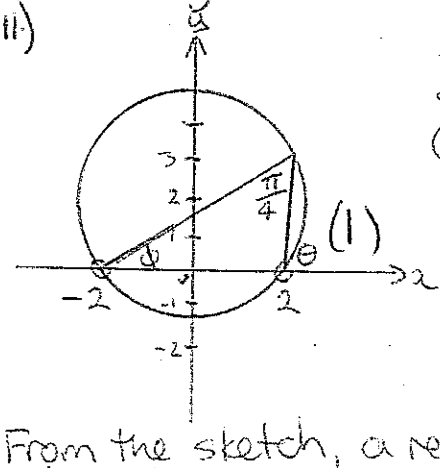
$\dfrac{y}{x-2} = \dfrac{x+2+y}{x+2-y}$

$\cancel{xy} + 2y - y^2 = (x^2 - 4) + \cancel{xy} - 2y$

$x^2 + y^2 - 4y - 4 = 0$

$x^2 + (y-2)^2 = 8$ (1)

ii.)



(1)

Consider the sketch centre $(0,2)$, radius $2\sqrt{2}$

From the sketch, a restriction is put on the locus and its equation becomes $x^2 + (y-2)^2 = 8$, $y > 0$
(1)

2d) i) $2\sqrt{3} - 2i = r\,\text{cis}\,\theta$

Then $r = \sqrt{(2\sqrt{3})^2 + (-2)^2}$

$= \sqrt{16}$

$= 4$

$\tan\theta = \dfrac{-2}{2\sqrt{3}}$

$= -\dfrac{1}{\sqrt{3}}$

$\therefore \theta = -\dfrac{\pi}{6}$

$\therefore 2\sqrt{3} - 2i = 4\,\text{cis}\left(-\dfrac{\pi}{6}\right)$ (1)

ii) $(2\sqrt{3} - 2i)^7 = \left[4\,\text{cis}\left(-\dfrac{\pi}{6}\right)\right]^7$

$= 4^7\,\text{cis}\left(-\dfrac{7\pi}{6}\right)$

$= 16384\,\text{cis}\left(\dfrac{5\pi}{6}\right)$ (1)

$= 16384\left[\cos\left(\dfrac{5\pi}{6}\right) + i\sin\left(\dfrac{5\pi}{6}\right)\right]$

$= 16384\left[-\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right]$ (1)

$= -8192\sqrt{3} + 8192i$ (1)

QUESTION 3

a) $P(x) = x^3 - 12x^2 + 36x + c$

$P'(x) = 3x^2 - 24x + 36$

$= 3(x-6)(x-2)$

$\therefore P'(x) = 0$ for $x = 6, 2$ (1)

So $P'(2) = P(2) = 0$

Hence $P(2) = 2^3 - 12(2)^2 + 36(2) + c = 0$

$\therefore 8 - 48 + 72 = -c$

$\therefore c = -32$ (1)

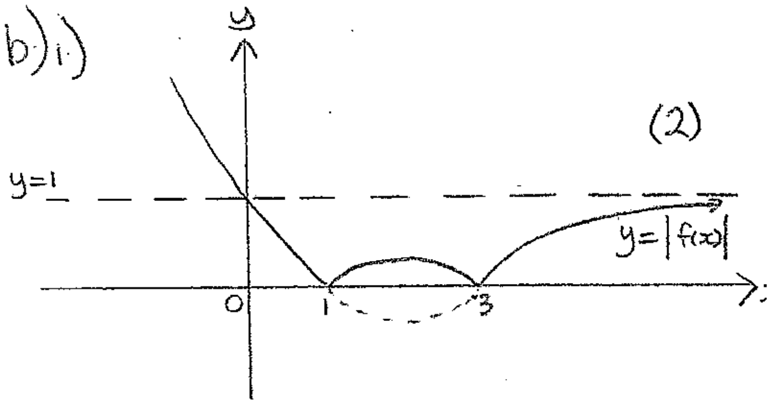
Also $P'(6) = P(6) = 0$

Hence $P(6) = 6^3 - 12(6^2) + 36(6) + c = 0$

$\therefore c = 0$ (1)

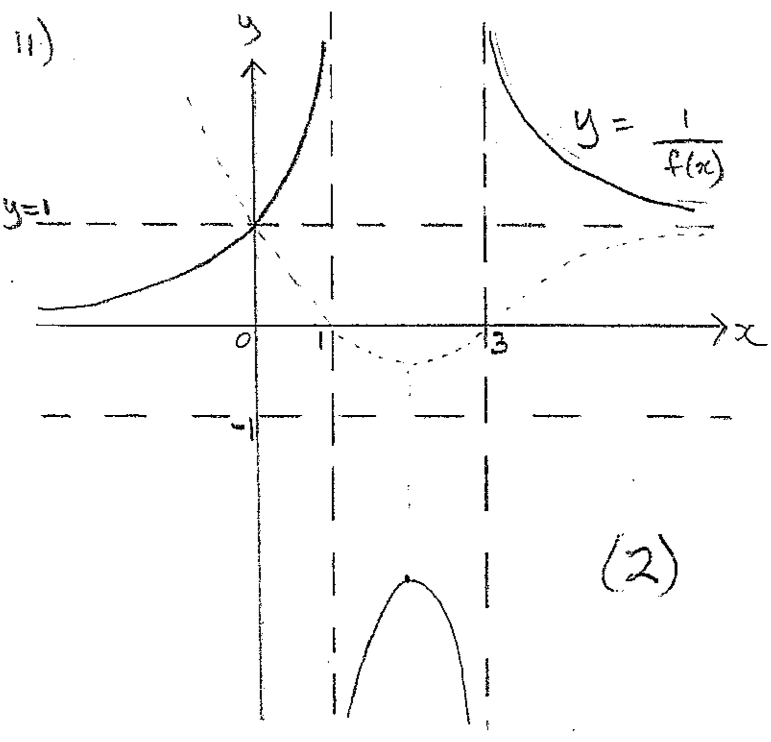
$\therefore P(x)$ has a double zero if and only if $c = -32$ or $c = 0$

b) i)



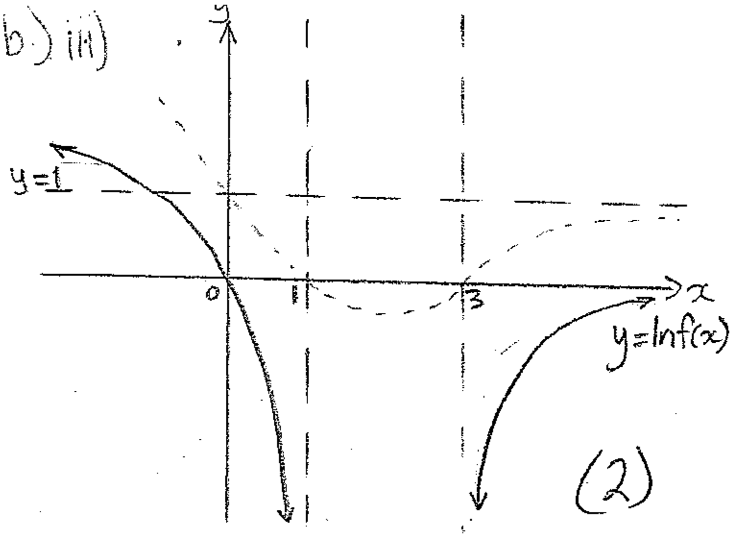
(2)

ii)



(2)

3b) iii)



(2)

c) $P(x) = x^4 - 2x^3 + 4x^2 - 3x + 1$

α, β, γ and δ are the roots of $P(x) = 0$

i) Only possible integer roots are ± 1

But $P(1) = 1 - 2 + 4 - 3 + 1$
$= -1$
$\neq 0$

$P(-1) = 1 + 2 + 4 + 3 + 1$
$= 11$
$\neq 0$ (1)

Hence there are no integer roots

ii) $P(x)$ is a continuous, real function and $P(0) = 1 > 0$ also
$P(1) = +1 > 0$ ∴ indeterminate.

~~Hence, considering the graph of $y = P(x)$, a real root exists between 0 and 1~~

iii) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$

$(\alpha + \beta + \gamma + \delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$ (1)

$= 2^2 - 2 \times 4$ (1)

$= -4$

iv) Since $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -4$, at least one of these squares must be negative.

Hence $P(x) = 0$ has a non-real root, then its complex conjugate is a second non-real root, since the coefficients of $P(x)$ are real. (1)

We know there is a real root between 0 and 1. Since the non-real roots come in complex conjugate pairs, the remaining fourth root cannot be non-real. (1)

Hence the equation $P(x) = 0$ has two real roots and two non-real roots.

Question 4

a) $y^3 + 2xy + x^2 + 2 = 0$

i) $3y^2 \dfrac{dy}{dx} + 2\left(1 \cdot y + x\dfrac{dy}{dx}\right) + 2x = 0$ (1)

$\dfrac{dy}{dx}(3y^2 + 2x) = -2(y + x)$ (1)

$\dfrac{dy}{dx} = \dfrac{-2(y+x)}{3y^2 + 2x}$ (1)

ii) $\dfrac{dy}{dx} = 0$ for $y = -x$ and

$y^3 + 2xy + x^2 + 2 = 0$

(sub in $y = -x$) (1)

$-x^3 - 2x^2 + x^2 + 2 = 0$

$x^3 + x^2 - 2 = 0$

$(x-1)(x^2 + 2x + 2) = 0$

Hence $(1, -1)$ is the only stationary point since quadratic factor has $\Delta < 0$ (1)

4b) i.)

$P(a\cos\theta, b\sin\theta)$, $Q(a\cos\phi, b\sin\phi)$ ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

gradient of PQ is

$$\frac{y_2-y_1}{x_2-x_1} = \frac{b(\sin\theta-\sin\phi)}{a(\cos\theta-\cos\phi)} = \frac{b}{a}\cdot\frac{2\sin\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)}{-2\sin\left(\frac{\theta-\phi}{2}\right)\sin\left(\frac{\theta+\phi}{2}\right)}$$

$$= -\frac{b}{a}\,\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\theta+\phi}{2}\right)}$$

$\therefore$ the equation of chord PQ is

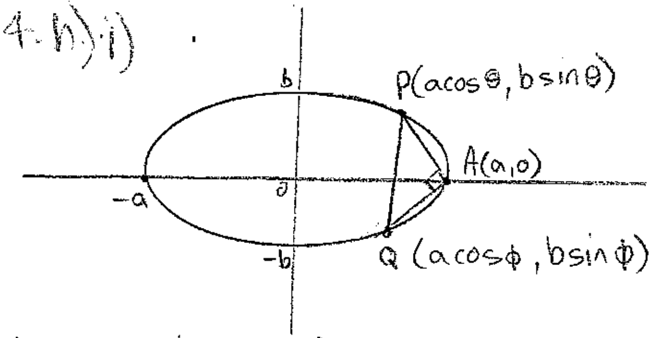
$$y-b\sin\theta = -\frac{b}{a}\,\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\sin\left(\frac{\theta+\phi}{2}\right)}(x-a\cos\theta)$$

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\theta\cos\left(\frac{\theta+\phi}{2}\right) + \sin\theta\sin\left(\frac{\theta+\phi}{2}\right)$$

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

where P, Q have parameters θ, ϕ.

4. h) i)



The chord PQ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has the eq^n

$$\frac{x}{a}\cos\left(\frac{\theta+\phi}{2}\right)+\frac{y}{b}\sin\left(\frac{\theta+\phi}{2}\right)=\cos\left(\frac{\theta-\phi}{2}\right)$$

where P, Q have parameters θ, φ. (1)

ii) The chord PQ cuts the x-axis at point T(t, 0).

$$t = a\cos\left(\frac{\theta-\phi}{2}\right)\sec\left(\frac{\theta+\phi}{2}\right)$$ (1)

$$= a\left(1+\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)\left(1-\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)^{-1}$$

The gradient AP is $\frac{b\sin\theta}{a(\cos\theta-1)}$

$$= -\frac{b}{a}\cot\frac{\theta}{2}$$

and gradient AQ is $\frac{b\sin\phi}{a(\cos\phi-1)}$

$$= -\frac{b}{a}\cot\frac{\phi}{2}$$

If the chord PQ subtends a right angle at the point A, then AP × gradient AQ = −1.

Therefore

$$\frac{b^2}{a^2}\cot\frac{\theta}{2}\cot\frac{\phi}{2}=-1$$

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2}=-\frac{b^2}{a^2}$$ (1)

Hence $t = a\left(1-\frac{b^2}{a^2}\right)\left(1+\frac{b^2}{a^2}\right)^{-1}$

$$= a\left(\frac{a^2-b^2}{a^2+b^2}\right)$$ (1)

But for the ellipse

$b^2 = a^2(1-e^2)$

thus $t = \frac{ae^2}{2+e^2}$ (1)

So PQ passes through a fixed point $T\left(\frac{ae^2}{2+e^2}, 0\right)$ on the x-axis.

[4]

4) c) $\int \sin^{-1} x \, dx$ $\quad u = x \quad v = \sin^{-1} x$, $\quad u' = 1 \quad v' = \frac{1}{\sqrt{1-x^2}}$

$= x\sin^{-1}x - \int x \times \frac{1}{\sqrt{1-x^2}}\, dx$

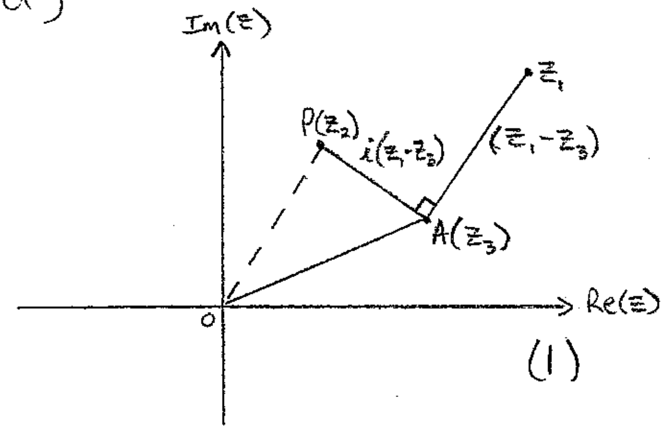
$= x\sin^{-1}x - \int x(1-x^2)^{-\frac{1}{2}} dx$ (1)

$= x\sin^{-1}x + \frac{1}{2}\int -2x(1-x^2)^{-\frac{1}{2}} dx$

$= x\sin^{-1}x + \frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= x\sin^{-1}x + \sqrt{1-x^2} + C$ (1)

d)



(1)

$\overrightarrow{AZ_1} = z_1 - z_3$

$\overrightarrow{AP} = i(z_1 - z_3)$

$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$ (1)

$\therefore OP = z_3 + i(z_1 - z_3)$

$= z_3 + iz_1 - iz_3$ (1)

$= (1-i)z_3 + iz_1$

Question 5

a) Let $S(n)$, $n = 2, 3, \dots$ be the sequence defined by

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$

Consider $S(2)$:

$\frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4} < \frac{6}{4} = 2 - \frac{1}{2}$

Hence true for $S(2)$ (1)

Assume true for $S(k)$:

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} < 2 - \frac{1}{k}$ (1)

Show true for $S(k+1)$ if true for $S(k)$:

ie $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$

$RHS = 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$

$= 2 - \frac{1}{k+1}\left(\frac{k+1}{k} - \frac{1}{k+1}\right)$

$= 2 - \frac{1}{k+1}\left(\frac{(k+1)^2 - k}{k(k+1)}\right)$

$= 2 - \frac{1}{k+1}\left(\frac{k^2+2k+1-k}{k(k+1)}\right)$

$= 2 - \frac{1}{k+1}\left(\frac{k^2+k+1}{k(k+1)}\right)$

$= 2 - \frac{1}{k+1}\left(\frac{k(k+1)}{k(k+1)} + \frac{1}{k(k+1)}\right)$

$= 2 - \frac{1}{k+1}\left(1 + \frac{1}{k(k+1)}\right)$

$< 2 - \frac{1}{k+1}$ since $k(k+1) > 0$

(1)

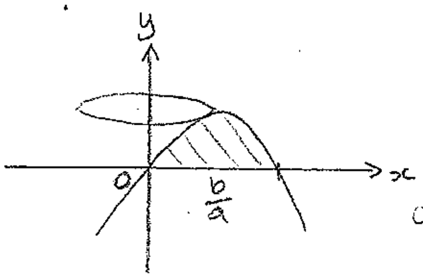
$\therefore LHS < 2 - \frac{1}{k+1}$

$\therefore S(k+1)$ is true if $S(k)$ is true. (1

But $S(2)$ is true $\therefore S(3)$ is true...

Hence $S(n)$ is true for $n = 2, 3, 4, \dots$

5.) b.)



$y = bx - ax^2$
$y = x(b - ax)$
$a > 0,\ b > 0$

$$V = \int_0^{\frac{b}{a}} 2\pi xy\,dx$$

$$= \int_0^{\frac{b}{a}} 2\pi \cdot x \cdot (bx - ax^2)\,dx \quad (1)$$

$$= 2\pi \int_0^{\frac{b}{a}} (bx^2 - ax^3)\,dx$$

$$= 2\pi \left[\frac{bx^3}{3} - \frac{ax^4}{4}\right]_0^{\frac{b}{a}} \quad (1)$$

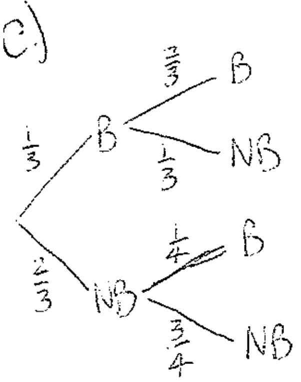
$$= 2\pi \left[\frac{b}{3} \times \frac{b^3}{a^3} - \frac{a}{4} \times \frac{b^4}{a^4} - 0\right]$$

$$= 2\pi \left[\frac{b^4}{3a^3} - \frac{b^4}{4a^3}\right]$$

$$= 2\pi \frac{b^4}{12a^3}$$

$$= \frac{\pi b^4}{6a^3} \text{ cubic units} \quad (1)$$

c.)



i) P(B on second)

$$= \frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{4} \quad (1)$$

$$= \frac{2}{9} + \frac{1}{6}$$

$$= \frac{7}{18} \quad (1)$$

ii) P(Misses on 3rd) =

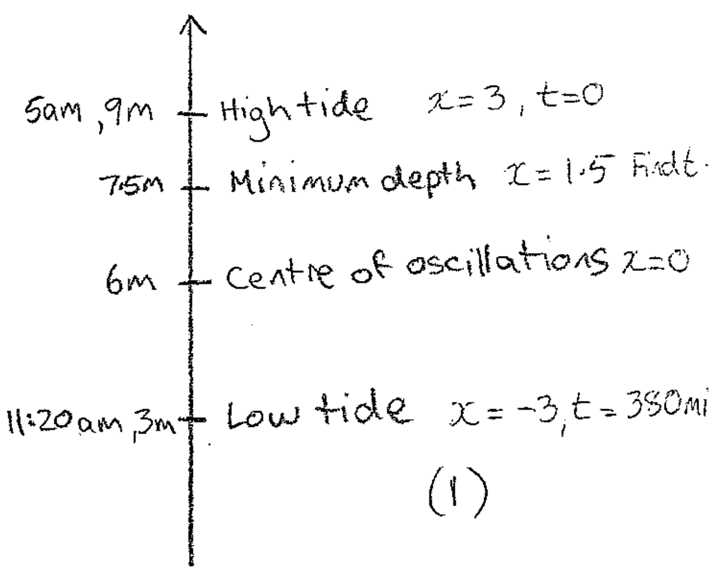
P(B, B, NB) + P(B, NB, NB) + P(NB, NB, NB) + P(NB, B, N[illegible]

$$= \frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} \times \frac{3}{4} + \frac{2}{3} \times \frac{3}{4} \times \frac{3}{4} + \frac{2}{3} \times \frac{1}{4} \times \frac{1}{3} \quad (1)$$

$$= \frac{2}{27} + \frac{1}{12} + \frac{3}{8} + \frac{1}{18}$$

$$= \frac{127}{216} \text{ or } 0.58796 \quad (1)$$

d.)



Period $T = 2(11{:}20 - 5) = 760$ minutes.

Amplitude $= \frac{1}{2}(9 - 3) = 3$m

Motion is SHM: $\ddot{x} = -n^2 x$

$$n = \frac{2\pi}{T} = \frac{\pi}{380} \quad (1)$$

$\therefore x = 3\cos(nt + \alpha)$, $0 \le \alpha \le 2\pi$

Initial conditions: $t = 0$, $x = 3$, $\cos\alpha = 1$

$\alpha = 0 \ \therefore \ x = 3\cos nt$. (1)

A minimum depth is 7.5m if $x = 1.5$

$$1.5 = 3\cos\left(\frac{\pi}{380} t\right)$$

$$\frac{1}{2} = \cos\left(\frac{\pi}{380} t\right)$$

$$\frac{\pi t}{380} = \frac{\pi}{3}$$

$$t = \frac{380}{3} = 2.06$$

Hence the latest time before noon when a minimum depth of 7.5m of water is $5 + 2.06 = 7.06$am. (1)

Question 6

a) i) $b^2 = a^2(e^2-1)$

$e^2 = 1 + \frac{b^2}{a^2}$

$= 1 + \frac{128}{16}$

$= 9$

$\therefore e = 3$ (1)

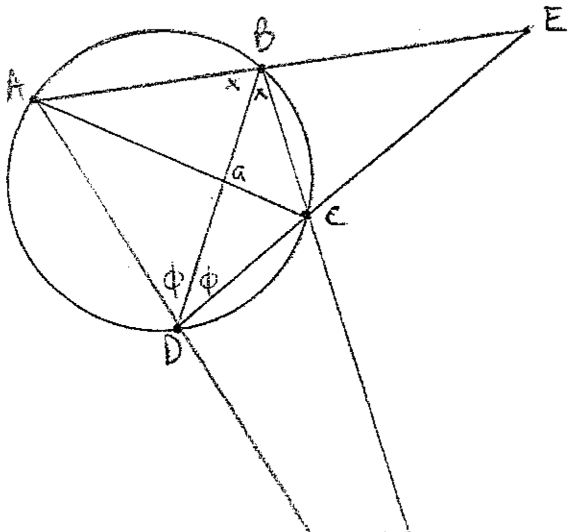
ii) Foci have coordinates $(\pm ae, 0)$

$\therefore (9,0)$ and $(-9,0)$ (1)

iii) Directrices have equations $x = \pm \frac{a}{e}$

$\therefore x = \frac{4}{3}$ and $x = -\frac{4}{3}$ (1)

b)



) $\angle ABD = \angle DBC = x$ (Given)

$\angle BDA = \angle BDC = \phi$ (Given)

$\therefore \angle BAD + x + \phi = 180°$ (angle sum of $\Delta = 180°$) (1)

But $\angle ABC + \angle ADC = 180°$ (opp $\angle$s cyclic quad are supp.) (1)

$\therefore 2x + 2\phi = 180°$

$\therefore x + \phi = 90°$

F

$\therefore \angle BAD + 90° = 180°$ (1)

$\therefore \angle BAD = 90°$

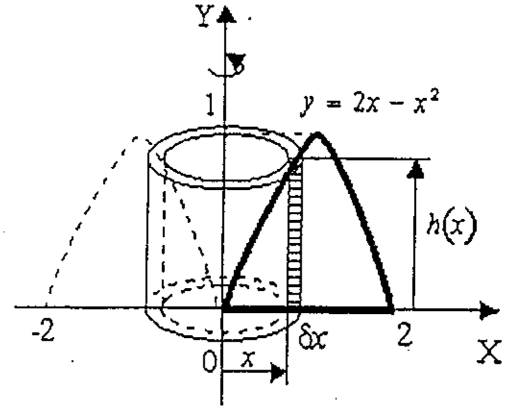
i) $\angle EAF = \angle BAD = 90°$

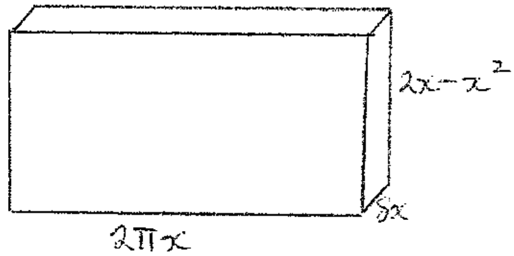
$\angle BCD = 90°$ since $\angle BAD + \angle BCD = 180°$ (cyclic quad) (1)

$\therefore \angle ECF = \angle BCD$ (vert opp $\angle$s equal) (1)

$\therefore \angle EAF = \angle ECF = 90°$ (1)

c)





Take a typical cylindrical shell of height $2x - x^2$

inner radius x

outer radius $x + \delta x$

The shell has volume

$\delta V = \pi\left[(x+\delta x)^2 - x^2\right](2x - x^2)$ (1)

$= 2\pi x(2x - x^2)\,\delta x$ (ignoring $(\delta x)^2$)

$\therefore$ Total $V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi x(2x - x^2)\,\delta x$ (1)

$$= 2\pi \int_0^2 x(2x - x^2)\,dx$$

$$= 2\pi \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2$$

$$= 2\pi \left[\left(\frac{16}{3} - 4\right) - 0\right]$$

$$= \frac{8\pi}{3} \text{ cubic units}$$ (1)

6) d) $I = \int_0^{\frac{\pi}{2}} e^x \cos x \, dx$

$$= \left[e^x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} e^x \sin x \, dx \quad (1)$$

$$= \left(e^{\frac{\pi}{2}} \cos \tfrac{\pi}{2} - e^0 \cos 0\right) + \int_0^{\frac{\pi}{2}} e^x \sin x \, dx$$

$$= -1 + \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx \quad (1)$$

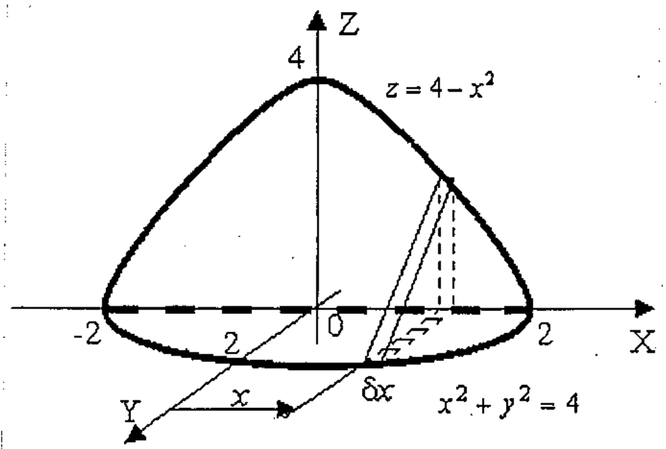
$$= -1 + e^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x \, dx$$

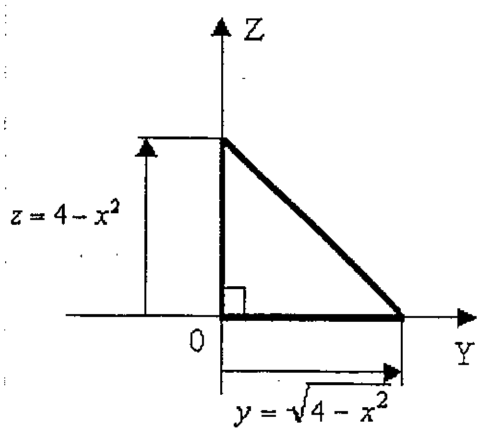
$$\therefore \; 2\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = e^{\frac{\pi}{2}} - 1$$

so $\int_0^{\frac{\pi}{2}} e^x \cos x \, dx = \frac{1}{2}\left(e^{\frac{\pi}{2}} - 1\right)$ (1)

Question 7

a)





i) The slice is a right-angled triangle with area of cross-section A, thickness δx.

$$A(x) = \frac{yz}{2}$$

$$= \frac{(4 - x^2)^{\frac{3}{2}}}{2} \quad (1)$$

The slice has volume

$$\delta V = A(x)\,\delta x$$

$$= \frac{(4 - x^2)^{\frac{3}{2}}}{2}\,\delta x \quad (1)$$

The volume of the solid is

$$V = \lim_{\delta x \to 0} \sum_{x=-2}^{2} \frac{(4 - x^2)^{\frac{3}{2}}}{2}\,\delta x$$

$$= \frac{1}{2}\int_{-2}^{2} (4 - x^2)^{\frac{3}{2}} \, dx \quad (1)$$

ii) Let $x = 2\sin\phi$ | $x = -2,\ \phi = 0$

$dx = 2\cos\phi \, d\phi$ | $x = 2,\ \phi = \frac{\pi}{2}$

$$V = 16\int_0^{\frac{\pi}{2}} (1 - \sin^2\phi)^{\frac{3}{2}} \cos\phi \, d\phi \quad (1)$$

$$= 16\int_0^{\frac{\pi}{2}} \cos^4\phi \, d\phi$$

$$= 16\int_0^{\frac{\pi}{2}} \left[\tfrac{1}{2}\left(1 + \cos 2\phi\right)\right]^2 d\phi$$

$$= 4\int_0^{\frac{\pi}{2}} \left[1 + 2\cos 2\phi + \tfrac{1}{2}\left(1 + \cos 4\phi\right)\right] d\phi \quad (1)$$

$$= 4\left[\tfrac{3}{2}\phi + \sin 2\phi + \frac{\sin 4\phi}{8}\right]_0^{\frac{\pi}{2}}$$

$$= 3\pi \quad (1)$$

$\therefore$ the volume of the solid is 3π cubic units.

1) b).i)

$$\left(a+\frac{1}{a}\right)^2=\left(a-\frac{1}{a}\right)^2+4\geq 4$$

since $\left(a-\frac{1}{a}\right)$ real

then $\left(a-\frac{1}{a}\right)^2 \geq 0$

$\therefore \left(a+\frac{1}{a}\right)\geq 2$ for $a>0$ (1)

ii.) $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)$

$= 1+1+\frac{a}{b}+\frac{b}{a}$

$= 2+\frac{a}{b}+\frac{b}{a} \geq 4,$

using part i.) with $a \to \frac{a}{b}$ (1)

$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

$= 1+1+1+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)$

$\geq 3+3\times 2$

$= 9$ (1)

iii.) $(a_1+a_2+\dots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)$

$= \sum_{i=1}^{n}\frac{a_i}{a_i}+\sum_{\substack{i=1\\ j>i}}^{n}\left(\frac{a_i}{a_j}+\frac{a_j}{a_i}\right)$ (1)

$= n\times 1+\sum_{\substack{i=1\\ j>i}}^{n}\left(\frac{a_i}{a_j}+\frac{a_j}{a_i}\right)$

There are ${}^nC_2$ ways of selecting two different integers from $1,2,3,\dots,n$. Hence there are ${}^nC_2$ terms of the form $\frac{a_i}{a_j}+\frac{a_j}{a_i}$ where $i<j$.

$\therefore (a_1+a_2+\dots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}\right)$

$\geq n+{}^nC_2\times 2$

$= n+n(n-1)$

$= n^2$ (1)

c.) i) $\frac{q}{p}=\frac{r}{q}$

$pr = q^2 > 0$

Hence p and r must have the same sign, and P and R must lie on the same branch. Q will lie on the opposite branch if the common ratio of the GP is negative. (1)

ii.) Gradient of PR is

$$\frac{c\left(\frac{1}{r}-\frac{1}{p}\right)}{c(r-p)}=\frac{p-r}{pr(r-p)}=\frac{-1}{pr} \text{ (1)}$$

$y=\frac{c}{t}$ $\qquad x=ct$

$\frac{dy}{dt}=\frac{-c}{t^2}$ $\qquad \frac{dx}{dt}=c$

$\therefore \frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$ (1)

$= -\frac{1}{t^2}$

$\therefore$ tangent at Q has gradient $\frac{-1}{q^2}=\frac{-1}{pr}$ (1)

Hence PR is parallel to tangent at Q.

Question 5

a) i) $\frac{d}{dx}\left[(1+x)\ln(1+x)-x\right]$

$= 1\cdot\ln(1+x)+(1+x)\cdot\frac{1}{1+x}-1$

$= \ln(1+x)$ (1)

$\therefore \int \ln(1+x)\,dx = (1+x)\ln(1+x)-x+C$

ii) $I_n=\int_0^1 x^n\ln(1+x)\,dx,\ n=0,1,2,\dots$

$=\left[x^n\left((1+x)\ln(1+x)-x\right)\right]_0^1$

$-n\int_0^1 x^{n-1}\left[(1+x)\ln(1+x)-x\right]dx,$

$n=1,2,\dots$ (1)

$=2\ln 2-1-n\int_0^1\left[x^{n-1}\ln(1+x)+x^n\ln(1+x)-\boxed{x^n}\right]dx$

$=2\ln2-1-nI_{n-1}-nI_n+\frac{n}{n+1}\left[x^{n+1}\right]_0^1$

$=2\ln2-1-nI_{n-1}-nI_n+\frac{n}{n+1}$

$=2\ln2-nI_{n-1}-nI_n+\frac{1}{n+1}$ (1)

$\therefore (n+1)I_n=2\ln2-\frac{1}{n+1}-nI_{n-1},$

$n=1,2,\dots$

iii) $I_0=\int_0^1\ln(1+x)\,dx$

$=\left[(1+x)\ln(1+x)-x\right]_0^1$

$=2\ln2-1$

$3I_2=2\ln2-\frac{1}{3}-2I_1$

$=2\ln2-\frac{1}{3}-(2\ln2-\frac{1}{2}-I_0)$

$\therefore 3I_2=(2\ln2-\frac{1}{3})-(2\ln2-\frac{1}{2})+(2\ln2-1)$

$=2\ln2-(1-\frac{1}{2}+\frac{1}{3})$

$=2\ln2-\frac{5}{6}$ (1)

$4I_3=2\ln2-\frac{1}{4}-3I_2$

$\therefore 4I_3=(2\ln2-\frac{1}{4})-(2\ln2-\frac{1}{3})$
$+(2\ln2-\frac{1}{2})+(2\ln2-1)$

$=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}$

$=\frac{7}{12}$ (1)

iv) $(n+1)I_n=(2\ln2-\frac{1}{n+1})-(2\ln2-\frac{1}{n})$
$+\dots+(-1)^{n-1}(2\ln2-\frac{1}{2})+(-1)^n(2\ln2-\frac{1}{1}),$

with $n+1$ terms

Hence if n odd, there is an even number of terms

$2\ln2-2\ln2+2\ln2-2\ln2+\dots=0$

and $(n+1)I_n=-\frac{1}{n+1}+\frac{1}{n}+\dots-\frac{1}{2}+\frac{1}{1}$

$=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots-\frac{1}{n+1}$. (1)

Also if n is even there is an odd number of terms

$2\ln2-2\ln2+2\ln2-2\ln2+2\ln2-\dots=2\ln2$

and $(n+1)I_n=2\ln2-\frac{1}{n+1}+\frac{1}{n}-\dots+\frac{1}{2}-\frac{1}{1}$

$=2\ln2-\left(\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots+\frac{1}{n+1}\right)$ (1)

b) i) $\tan4x=\frac{2\tan2x}{1-\tan^2 2x}$

$=\frac{2\left(\frac{2t}{1-t^2}\right)}{1-\left(\frac{2t}{1-t^2}\right)^2}$ where $t=\tan x$ (1)

$=\frac{\frac{4t}{1-t^2}}{\frac{(1-t^2)^2-(2t)^2}{(1-t^2)^2}}$ (1)

$=\frac{4t(1-t^2)}{t^4-6t^2+1}$

8.) b.) ii.) $\tan x \tan 4x = 1$

$$\frac{t\,(4t)(1-t^2)}{t^4-6t^2+1} = 1 \quad \text{from part (i)}$$

$$4t^2(1-t^2) = t^4-6t^2+1$$

$$4t^2-4t^4 = t^4-6t^2+1 \qquad (1)$$

$$5t^4-10t^2+1=0$$

iii.) $x = 18^\circ \quad \tan x \tan 4x = \tan 18^\circ \tan 72^\circ$

$$= \tan 18^\circ \cot 18^\circ$$

$$= 1 \qquad (1)$$

$x = 54^\circ \quad \tan x \tan 4x$

$$= \tan 54^\circ \tan 216^\circ$$

$$= \tan 54^\circ \tan 36^\circ$$

$$= \tan 54^\circ \cot 54^\circ$$

$$= 1 \qquad (1)$$

iv.) $5t^4 - 10t^2 + 1 = 0$

$$t^2 = \frac{10 \pm \sqrt{100-20}}{10}$$

$$= \frac{5 \pm 2\sqrt{5}}{5} \qquad (1)$$

Now the roots of the above equation are $\pm \tan 18^\circ$ and $\pm \tan 54^\circ$ and since $\tan 18^\circ < \tan 54^\circ$ (1)

then $\tan 54^\circ = \sqrt{\frac{5+2\sqrt{5}}{5}}$ (1)